Name: ______________________ **ID (helpful but not necessary):** ______________________

Clear your desk of everything except pens, pencils and erasers. Show all work clearly and in order. No notes, phones and calculators. You have 50 minutes to finish the 6 pages for 100 points.

1. (20 points) Check that whether the following statements are True or False. (a-f: 3 points; g: 2 points).

(a) True or False. The following linear system is inconsistent since geometrically the two equations represent two straight lines that do not intersect each other.

$$2x_1 - x_2 = 3$$
$$-4x_1 + 2x_2 = -6$$

The two equations represent the same line. There are infinitely many points on the line. The system has infinitely many solutions, therefore, is consistent.

(b) True or False. The following matrix is in its reduced row echelon form.

$$\begin{bmatrix} 1 & -3 & 0 & 2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(c) True or False. If

$$A = \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -2 & 1 \\ 0 & 4 \end{bmatrix}$$

then

$1\cdot(-2)+3\cdot 0=-2$

$$AB = \begin{bmatrix} -4 & 18 \\ 2 & 13 \end{bmatrix}$$

(d) True or False. It is not possible to multiply the first matrix times second because their dimensions (sizes) are not the same.

2×3 3×2 $\longrightarrow$ 2×2 matrix.

$$\begin{bmatrix} 3 & 5 & 1 \\ -2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \\ 4 & 1 \end{bmatrix} = \begin{bmatrix} 3\cdot 2+5\cdot 1+1\cdot 4 & 3\cdot 1+5\cdot 3+1\cdot 1 \\ -2\cdot 2+0+2\cdot 4 & -2\cdot 1+0+2\cdot 1 \end{bmatrix}$$

(e) True or False. Let A, B be two $n \times n$ matrices, then $(AB)^T = \cancel{A^T B^T}$. $B^T \cdot A^T$

(f) True or False. Let C be a nonsymmetric $n \times n$ matrix, then the following matrix is symmetric.

$$F = (I + C^T)(I + C) = I + C^T + C + C^T C$$

Therefore, $F^T = I^T + (C^T)^T + C^T + (C^T C)^T$
$= I + C + C^T + C^T C = F$.

(g) True or False. A matrix A is said to be skew symmetric if $A^T = -A$. According to the definition $A = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ is skew symmetric .

$$A^T = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \neq \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = -A$$

2. (20 points) Consider the following linear systems.

$$\begin{aligned} 2x_1 + x_2 + 3x_3 &= 0 \\ 4x_1 + 3x_2 + 5x_3 &= 1 \\ 6x_1 + 5x_2 + 5x_3 &= -3 \end{aligned}$$

4pt **(a)** Find the augmented matrix of the linear system.

$$\left[\begin{array}{ccc:c} 2 & 1 & 3 & 0 \\ 4 & 3 & 5 & 1 \\ 6 & 5 & 5 & -3 \end{array}\right]$$

2pt.

16pt **(b)** Solve the system if it is consistent or state why it is inconsistent. Show your work.

$$\longrightarrow \left[\begin{array}{ccc:c} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 2 & -4 & -3 \end{array}\right] \begin{array}{l} \text{Row②}-2\times\text{Row①} \\ \text{Row③}-3\cdot\text{Row①}. \end{array}$$

3pt/16

$$\longrightarrow \left[\begin{array}{ccc:c} 2 & 1 & 3 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & -5 \end{array}\right] \text{Row③}-2\cdot\text{Row②}.$$

3pt/16

The reduced system.

$$\begin{cases} 2x_1 + x_2 + 3x_3 = 0 \\ \quad\ \ x_2 - x_3 = 1 \\ \quad\quad -2x_3 = -5. \end{cases}$$

1pt

Back substitution:

$-2x_3 = -5 \Rightarrow x_3 = \frac{5}{2}$

3pt.

$x_2 - x_3 = 1 \Rightarrow x_2 = x_3 + 1 = \frac{5}{2} + 1 = \frac{7}{2}$

3pt

$2x_1 + x_2 + 3x_3 = 0 \Rightarrow 2x_1 = -x_2 - 3x_3 = -\frac{7}{2} - 3\cdot\frac{5}{2} = -11.$

$x_1 = -\frac{11}{2}$

3pt

The solution is $x_1 = -\frac{11}{2}$, $x_2 = \frac{7}{2}$, $x_3 = \frac{5}{2}$.

or $\left(-\frac{11}{2}, \frac{7}{2}, \frac{5}{2}\right)$.

3. (20 points) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix}$. Find A^{-1} and express A^{-1} as a product of elementary matrices.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right]$$ 4pt

$$\underline{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}} \rightarrow$$ 3pt

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$ 4pt

$$\underline{E_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \rightarrow$$ 3pt

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 2 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \end{array}\right]$$ 4pt.

$$A^{-1} = \begin{bmatrix} 2 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = E_2 \cdot E_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$ 2pt

(1 pt if $E_1 \cdot E_2$).

4. (20 points) let

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 4 & 1 & -2 \\ 2 & 1 & 3 \end{bmatrix}$$

16pt **(a)** Evaluate the determinant $\det(A)$. (You can compute either by the definition of the determinant directly or by applying the row operation to simply the matrix first.)

$$\det A = 1 \cdot \begin{vmatrix} 1 & -2 \\ 1 & 3 \end{vmatrix} - 3 \cdot \begin{vmatrix} 4 & -2 \\ 2 & 3 \end{vmatrix} + 2 \cdot \begin{vmatrix} 4 & 1 \\ 2 & 1 \end{vmatrix}$$

2pt 2pt 2pt

$$= 1 \cdot (3-(-2)) - 3 \cdot (4 \cdot 3 - (-2) \cdot 2) + 2 (4 \cdot -2)$$

2pt 2pt 2pt

$$= 1 \cdot 5 - 3 \cdot 16 + 2 \cdot 2.$$

1pt 1pt 1pt

$$= -39.$$

1pt

4pt **(b)** Use your answer to part (a) to find $\det(-A)$.
Hint: consider $-A = (-1)A$.

$$\det(-A) = (-1)^3 \cdot \det A$$

$$= -1 \cdot (-39) \quad \text{2pt}$$

$$= 39. \quad \text{2pt}$$

5. (10 points) Let

$$R = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}.$$

Prove that $R^{-1} = R^T$.

Formula: $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11} \cdot a_{22} - a_{12} \cdot a_{21}} \cdot \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$R^{-1} = \frac{1}{\underbrace{\frac{1}{2} \cdot \frac{1}{2} - \left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)}_{2pt.}} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad 3pt$$

$$= \frac{1}{\underbrace{\frac{1}{4} + \frac{3}{4}}_{2pt}} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$$

$$= \frac{1}{1} \cdot \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \quad 1pt.$$

Def. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$

$$R^T = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}^T = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}. \quad 2pt.$$

Therefore, $R^{-1} = R^T$

6. (10 points) Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

6pt **(a)** Prove that $A^n = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ for all positive integers $n \geq 3$.

$$A^2 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$ 2pt

$$A^3 = A^2 \cdot A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 2pt

Therefore, $A^4 = A^3 \cdot A = 0 \cdot A = 0$

Inductively, $A^n = A^{n-1} \cdot A = 0$ for all $n \geq 3$. 2pt.

4pt **(b)** Prove that $I - A$ is invertible and has the following inverse formula:

$$(I - A)^{-1} = I + A + A^2$$

Direct computation (and by $A^3 = 0$ in (a)):

$$(I-A)\cdot(I+A+A^2) = I + A + A^2 - A(I+A+A^2)$$
$$= I + \underbrace{A + A^2} - \underbrace{A - A^2} - A^3 = I - A^3 = I.$$ 2pt

$$(I+A+A^2)\cdot(I-A) = I + A + A^2 - (I+A+A^2)A$$
$$= I + \underbrace{A + A^2} - \underbrace{A - A^2} - A^3 = I - A^3 = I$$ 2pt.

Therefore, by the definition of the inverse, $(I-A)^{-1} = I + A + A^2$